MASSACHUSETTS INSTITUTE OF TECHNOLOGY
HAYSTACK OBSERVATORY
WESTFORD, MASSACHUSETTS 01886

June 20, 2006

To: SRT Group

From: A.E.E. Rogers

Subject: Modelling Sun's radio image using closure phase

The "closure phase" is the sum of 3 phases taken around a triangle of baselines in a manner that results in the cancellation of the instrumental phases and clock errors which are station dependent.

The closure phase ϕ_c is given by

$$\phi_c = \phi_{12} - \phi_{13} + \phi_{23}$$

where ϕ_{12} = phase on baseline 12
ϕ_{13} = phase on baseline 13
ϕ_{23} = phase on baseline 23

The phases are calculated by the program modeling source structure using amplitudes and closure phase.

The "source visibility" or complex fringe amplitude is given by

$$V_{12}(u,v) = \iint B(\alpha,\delta)e^{iu\alpha}e^{iv\delta}d\alpha d\delta$$

where $B(\alpha,\delta)$ is the source brightness distribution u,v are the projected baselines in the direction of the source in the right ascension and declination detection respectively.

The normalized visibility amplitude and phase

$$A_{12}(u,v) = |V_{12}(u,v)|/V_{12}(o,o)$$

$$\phi_{12}(u,v) = a\tan 2(\operatorname{Im}V, \operatorname{Re}V)$$

The normalized fringe amplitude $a_{12}(u,v)$ from vlbiproc is given by

$$a = \rho|V_{12}(u,v)|/V_{12}(o,o)$$

where $\rho = \left[\dfrac{T_{sun1}T_{sun2}}{(T_{sun1}+T_{s1})(T_{sun2}+T_{s2})}\right]^{\frac{1}{2}}$

The source structure can be modeled to fit the closure phase and amplitudes on the 3 baselines if only simple structures which can be characterized by a few parameters are considered. Full image reconstruction using the 2-D fourier transform from the spatial frequencies of the *(u,v)* plane to the brightness plane with subsequent "cleaning" is not practical owing to the extremely sparse sampling of the *(u,v)* plane.

For example, a simple model of the Sun might consist of a uniform disk plus a single unresolved "sunspot" with enhanced radio emission. In this case the model has 3 parameters.

        a. Relative flux of sunspot
        b. Position angle of the sunspot
        c. Radial distance of the sunspot from the center of the disk

Since the effects of these parameters on the normalized amplitude and phase is non linear the best model has to be found by searching through the parameter space to find the best fit. The criteria for the best fit is to minimize the sum of the residuals (residual equals observed quantity minus modeled quantity) squared. The "least squares" is equivalent to finding the most likely model (maximum likelihood) when the errors are Gaussian. Once a model is found that minimizes the sum of residuals squared and has residuals that follow the expected statistic of the noise in the data we have extracted all the available information in the data. We may be able to find more complex models which fit the data equally well (or better if the model starts to fit the noise) but these more complex models are not justified. In fact we need to find the simplest model to fit the data within the noise.

For least squares, the sum Q, has "chi-squared" statistics and is given by

$$Q = \sum_i w_i \left(m_i - a_i \right)^2 + \sum_j w_j \left(\cos\left(mc_j \right) - \cos\left(c_j \right) \right)^2 + \sum_j w_j \left(\sin\left(mc_j \right) - \sin\left(c_j \right) \right)^2$$

where m_i = model amplitudes
        a_i = observed amplitudes
        mc_j = model closure
        c_j = observed closure phases
        w = weight = $1/\sigma^2$

The search for a 3-parameter model requires a 3-D search which is only practical using efficient coding of the algorithm needed to obtain the visibility. For a uniform annulus and uniform disk the following integrals are useful

$$\frac{1}{2\pi} \int_o^{2\pi} e^{iz\sin\phi} d\phi = J_0\left(z \right)$$

and $\int_o^r z J_0\left(z \right) dz = r J_1\left(r \right)$

where J_0 and J_1 are Bessel functions of zero and first order respectively. Also it can be advantageous to separate the complete visibility into components which depend on parameters and those which are constant. For example in the example of disk plus a sunspot the contribution of the disk is constant and independent of the sunspot parameters.

The closure phase can only take on values of *0* or π for any source that has refection symmetry about a line through the centroid.

It is convenient to use radial coordinates, $\left(r, \theta \right)$ which calculating the visibility of the Sun.

For uniform disk

$$V(z) = \int\int_{o}^{R}\int_{o}^{2\pi} e^{irz\sin\theta} \, rd\theta dr \Big/ \pi R^2$$

$$= \int_{o}^{R} 2rJ_0(rz)dr \Big/ R^2$$

$$= 2J_1(Rz)/Rz$$

where z = interferometric phase radians per radian

$= 2\pi$ projected baseline in meters/wavelength in meters

R = radius of solar disk in radians.

(SRT memo 18) gives some of the geometry calculations). If the math library contains first order Bessel contains first order Bessel function this component can be efficiently calculated. Otherwise the visibility integral can be calculated in a double loop. If the sunspot is small the visibility is given by

$$v = ae^{iu\alpha}e^{iv\delta}$$

where a = strength of sunspot

α, δ = coordinates on the solar disk

u, v = projected baseline components

In imaging the Sun the normal convention is to display the image with coordinates referred to the Sun's rotation pole. This can be accomplished by first determining the position angle of the Sun's rotation pole, θ_s, and then rotating the projected baseline by this angle.

$$\theta_s = atan2\big(\cos(decp)\sin(rap - ras), \sin(decp)\cos(decs)\big)$$

rap = 286.11˚, decp = 63.85˚ is the *ra* and *dec* of the Sun's pole and *ras, decs* is the *ra* and *dec* of the Sun.

$$usun = u\cos(\theta_s) - v\sin(\theta_s)$$

$$vsun = v\cos(\theta_s) + u\sin(\theta_s)$$

Figure 1 shows a sample plot of a model fit to 3-baseline data taken 19 June 2006.

Sun0

1.00

Sun1

1.14

baseline 2

baseline 1

baseline 0

closure phase

15   16   17   18   19   20   21   22

UT time (hrs)
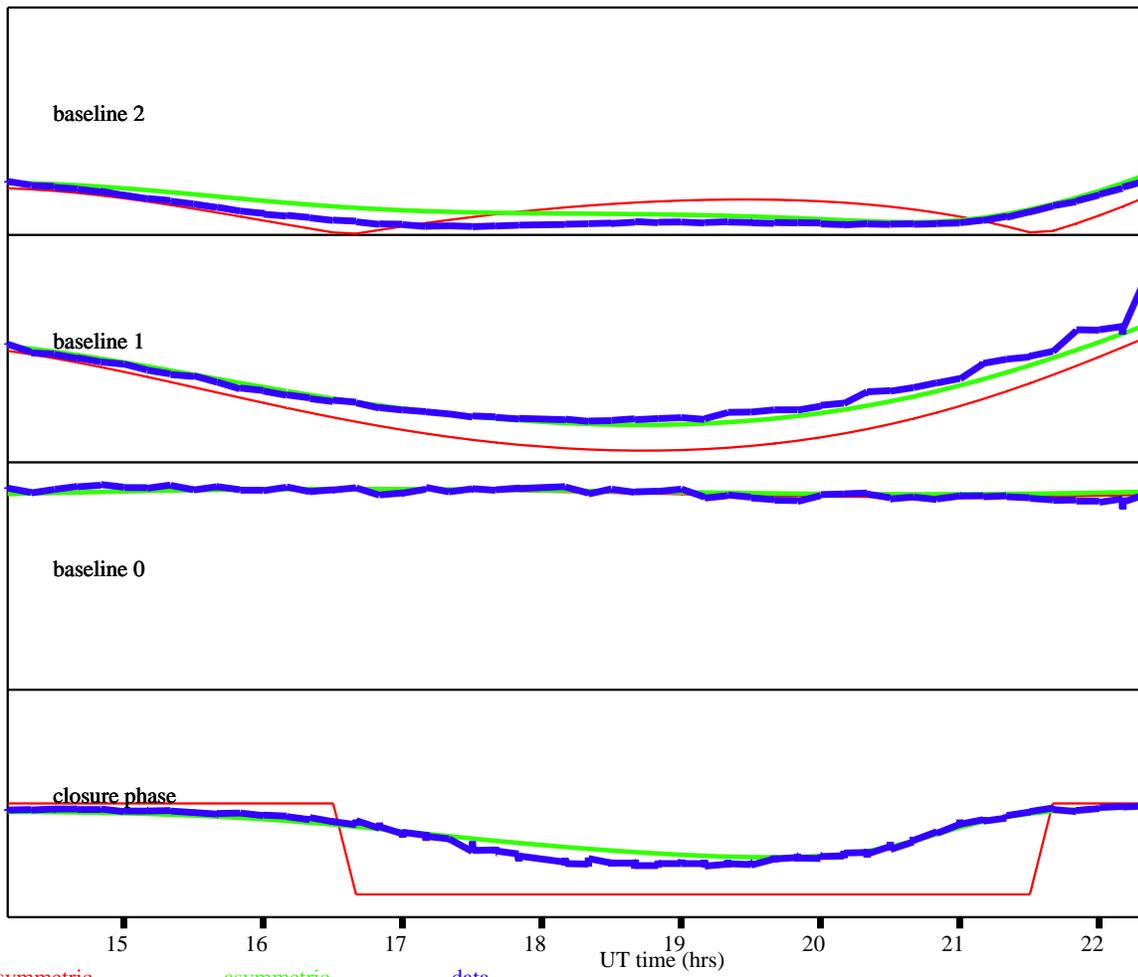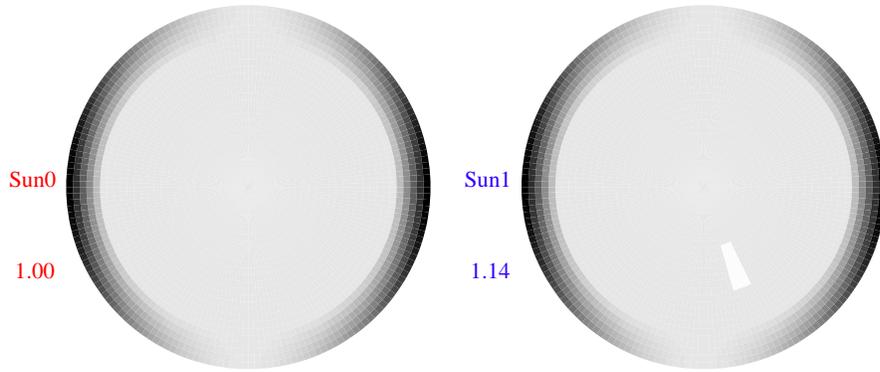
— symmetric   — asymmetric   — data

fit 0.11 baam 0.14 limratio 10.00

/data1/srt/2006_170_14_10.Haysrt2 /data1/srt/2006_170_14_10.Haysrt3

/data1/srt/2006_170_22_20.Haysrt2 /data1/srt/2006_170_22_20.Haysrt3